Complex number 2

1. Given $z_1 = 2 + i$, $z_2 = 3 - 4i$ and $\frac{1}{z_3} = \frac{1}{z_1} + 2z_2$, find z_3.

Write your answer in the standard form of $a + bi$.
Hence, find the modulus and principal argument of z_3.
State your answers correct to three decimal places.

$$\frac{1}{z_3} = \frac{1}{z_1} + 2z_2 \Longrightarrow z_3 = \frac{z_1}{1+2z_1z_2} = \frac{2+i}{1+2(2+i)(3-4i)} = \frac{2+i}{21-10i} = \frac{(2+i)(21+10i)}{(21-10i)(21+10i)} = \frac{32+41i}{541} = \frac{32}{541} + \frac{41}{541}i$$

$$|z_3| = \sqrt{\left(\frac{32}{541}\right)^2 + \left(\frac{41}{541}\right)^2} = \frac{\sqrt{2705}}{541} \approx 0.0961360711567 \approx \mathbf{0.096} \text{ (to 3 dec.places)}$$

$$\operatorname{Arg}(z_3) = \tan^{-1}\frac{\frac{41}{541}}{\frac{32}{541}} = \tan^{-1}\frac{41}{32} \approx 0.9080668189019 \approx \mathbf{0.908\ radians} \text{ (to 3 dec.places)}$$

2. Given that $|z - 1| = 2|z + 1|$, find the Cartesian equation of the locus of the point P representing complex number z. .

Hence, sketch the locus of the point P on an Argand diagram.

Let $z = x + yi$

$|z - 1| = 2|z + 1|$

$\Longrightarrow |x + yi - 1| = 2|x + yi + 1|$

$\Longrightarrow |(x - 1) + yi|^2 = 4|(x + 1) + yi|^2$

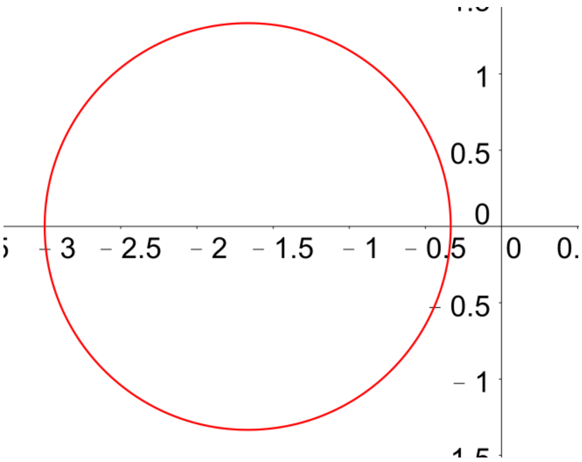
$\Longrightarrow (x - 1)^2 + y^2 = 4[(x + 1)^2 + y^2]$

$\Longrightarrow 3x^2 + 3y^2 + 10x + 3 = 0$

$\Longrightarrow x^2 + y^2 + \frac{10}{3}x + 1 = 0$

Locus is a circle centre $= -\frac{5}{3}$,

radius $= \sqrt{\left(-\frac{5}{3}\right)^2 - 1} = \frac{4}{3}$



3. Solve the equation $z^5 + 32\mathrm{i} = 0$.

$z^5 = -32\mathrm{i} = 32\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right) = 32\text{ cis }\left(2k\pi + \frac{\pi}{2}\right)$, where $k\epsilon\mathbf{Z}$.

By de Moirves' Theorem,

$z_k = (-32\mathrm{i})^{\frac{1}{5}} = \left[32\text{ cis }\left(2k\pi + \frac{\pi}{2}\right)\right]^{\frac{1}{5}} = 2\text{ cis }\left(\frac{2k\pi + \frac{\pi}{2}}{5}\right)$, where $k = 0,1,2,3,4$.

$z_0 = 2\text{ cis }\left(\frac{\pi}{10}\right) \approx 1.9021 - 0.6180i$

$z_1 = 2\text{ cis }\left(\frac{3\pi}{10}\right) \approx 1.1756 + 1.6180i$

$z_2 = 2\text{ cis }\left(\frac{5\pi}{10}\right) = -2i$

$z_3 = 2\text{ cis }\left(\frac{7\pi}{10}\right) \approx -1.1756 + 1.6180i$

$z_4 = 2\text{ cis }\left(\frac{9\pi}{10}\right) \approx -1.9021 - 0.6180i$

4. If the equation $\mathrm{z}^3 + \mathrm{az} + \mathrm{b} = 0$ has a root $\mathrm{z} = -1 + \mathrm{i}$ where a, b are real numbers, find the values of a, b. Show that $\mathrm{z} = -1 - \mathrm{i}$ is also a root of the equation.

$(-1 + \mathrm{i})^3 + \mathrm{a}(-1 + \mathrm{i}) + \mathrm{b} = 0 \Longrightarrow (2 + 2\mathrm{i}) + \mathrm{a}(-1 + \mathrm{i}) + \mathrm{b} = 0 \Longrightarrow (2 - \mathrm{a} + \mathrm{b})\mathrm{i} + (2 + \mathrm{a}) = 0$

$\begin{cases} 2 - \mathrm{a} + \mathrm{b} = 0 \\ 2 + \mathrm{a} = 0 \end{cases} \Longrightarrow \begin{cases} \mathrm{a} = -2 \\ \mathrm{b} = -4 \end{cases}$

The equation becomes $\mathrm{z}^3 - 2\mathrm{z} - 4 = 0$

Since $(-1 - \mathrm{i})^3 - 2(-1 - \mathrm{i}) - 4 = (2 - 2\mathrm{i}) - 2(-1 - \mathrm{i}) - 4 = 0$, $\mathrm{z} = -1 - \mathrm{i}$ is also a root of the equation.

5. **(a)** One of the roots of the equation $4\mathrm{x}^3 + \mathrm{x} + 5 = 0$ is an integer. Find this root and write down a quadratic equation for the remaining roots. Find these roots, expressing your answer in the satandard form of $\mathrm{a} + \mathrm{bi}$.

(b) By writing $\mathrm{y} = \frac{1}{\mathrm{x}}$, find the roots of the equation $5\mathrm{y}^3 + \mathrm{y}^2 + 4 = 0$,

giving the complex roots in the form $\mathrm{a} + \mathrm{bi}$.

(a) $f(x) = 4x^3 + x + 5$, $f(-1) = 4(-1) + (-1) + 5 = 0 \Longrightarrow (x+1)$ is a factor of $f(x)$.

By division, we have $4x^3 + x + 5 = (x+1)\,(4\,x^2 - 4\,x + 5) = 0$

$$\therefore x = -1 \text{ or } \tfrac{1}{2} - \text{i or } \tfrac{1}{2} + \text{i}.$$

(b) $5y^3 + y^2 + 4 = 0 \Longrightarrow 5\left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^2 + 4 = 0 \Longrightarrow 4x^3 + x + 5 = 0$

$$\therefore \frac{1}{y} = -1 \text{ or } \frac{1}{y} = \frac{1}{2} - \text{i or } \frac{1}{y} = \frac{1}{2} + \text{i}$$

$$\therefore y = -1 \text{ or } \frac{2}{5} + \frac{4}{5}\text{i or } \frac{2}{5} - \frac{4}{5}\text{i}$$

6. Find the roots of the equation $(z - i\alpha)^3 = i^3$, where α is a real constant.

(a) Show that the points representing the roots of the above equation form an equilateral triangle.

(b) Solve the equation $[z - (1+i)]^3 = (2i)^3$.

(c) If ω is a root of the equation $ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$, show that the conjugate ω' is also a root of this equation.

(d) Hence, or otherwise, obtain a polynomial equation of degree six with three of its roots also the roots of the equation $(z-1)^3 = i^3$

(a) $(z - i\alpha)^3 = i^3 = -i = \text{cis}\left(2k\pi + \frac{3\pi}{2}\right), k \in R$

$$z - i\alpha = \left[\text{cis}\left(2k\pi + \frac{3\pi}{2}\right)\right]^{1/3} = \text{cis}\left(\frac{2k\pi + \frac{3\pi}{2}}{3}\right), k = 0,1,2.$$

$$\therefore z = i\alpha + \text{cis}\left(\frac{2k\pi + \frac{3\pi}{2}}{3}\right), k = 0,1,2.$$

$$z_0 = i\alpha + \cos\left(\frac{0 + \frac{3\pi}{2}}{3}\right) + i\sin\left(\frac{0 + \frac{3\pi}{2}}{3}\right) = \boldsymbol{(\alpha + 1)i}$$

$$z_1 = i\alpha + \cos\left(\frac{2\pi + \frac{3\pi}{2}}{3}\right) + i\sin\left(\frac{2\pi + \frac{3\pi}{2}}{3}\right) = \boldsymbol{-\frac{\sqrt{3}}{2} + \left(\alpha - \frac{1}{2}\right)i}$$

$$z_2 = i\alpha + \cos\left(\frac{4\pi + \frac{3\pi}{2}}{3}\right) + i\sin\left(\frac{4\pi + \frac{3\pi}{2}}{3}\right) = \boldsymbol{\frac{\sqrt{3}}{2} + \left(\alpha - \frac{1}{2}\right)i}$$

$$|z_0 - z_1| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left[(\alpha+1) - \left(\alpha - \frac{1}{2}\right)\right]^2} = 3$$

$$|z_1 - z_2| = \sqrt{\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)^2 + \left[\left(\alpha - \frac{1}{2}\right) - \left(\alpha - \frac{1}{2}\right)\right]^2} = 3$$

$$|z_2 - z_0| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left[\left(\alpha - \frac{1}{2}\right) - (\alpha + 1)\right]^2} = 3$$

Thus the points representing the roots of the above equation form an equilateral triangle.

(b) $[z - (1 + i)]^3 = (2i)^3 \Rightarrow \left[\frac{z-(1+i)}{2}\right]^3 = i^3 \Rightarrow \left[\frac{z-1}{2} - \frac{1}{2}i\right]^3 = i^3$

By (a), $\quad \frac{z_0-1}{2} = \left(\frac{1}{2} + 1\right) i \Rightarrow z_0 = \mathbf{1 + 3i}$

$\frac{z_1-1}{2} = -\frac{\sqrt{3}}{2} + \left(\frac{1}{2} - \frac{1}{2}\right) i \Rightarrow z_1 = \mathbf{1 - \sqrt{3}}$

$\frac{z_2-1}{2} = \frac{\sqrt{3}}{2} + \left(\frac{1}{2} - \frac{1}{2}\right) i \Rightarrow z_2 = \mathbf{1 + \sqrt{3}}$

(c) If ω is a root of the equation $ax^2 + bx + c = 0$, then $a\omega^2 + b\omega + c = 0$

$\overline{a\omega^2 + b\omega + c} = \bar{0} \Rightarrow \bar{a}\overline{\omega^2} + \bar{b}\bar{\omega} + \bar{c} = 0 \Rightarrow a\bar{\omega}^2 + b\bar{\omega} + c = 0$, since $a, b, c \in R$

$\omega' = \bar{\omega}$ is also a root of this equation.

Alternatively, we can set $\omega = u + vi, \omega' = u - vi$

$$a\omega^2 + b\omega + c = 0 \Rightarrow a(u + vi)^2 + b(u + vi) + c = 0$$
$$\Rightarrow (a\,u^2 - a\,v^2 + bu + c) + (2auv + bv)i = 0$$
$$\Rightarrow (a\,u^2 - a\,v^2 + bu + c = 0) \text{and } (2auv + bv) = 0$$

$a\omega'^2 + b\omega' + c = a(u - vi)^2 + b(u - vi) + c = (a\,u^2 - a\,v^2 + bu + c) - (2auv + bv)i = 0$

Thus, ω' is also a root of the equation.

(d) Method 1 (More complicate, but it satisfies the former part of quadratics)

Replace $i\alpha$ by 1 in part (a), the roots of $(z - 1)^3 = i^3$ are

$z_0 = \mathbf{1 + i}$, $\quad z_1 = 1 - \frac{\sqrt{3}}{2} - \frac{1}{2}\mathbf{i}$, $\quad z_2 = \mathbf{1} + \frac{\sqrt{3}}{2} - \frac{1}{2}\mathbf{i}$

Their conjugates are $z_0' = \mathbf{1 - i}$, $\quad z_1' = 1 - \frac{\sqrt{3}}{2} + \frac{1}{2}\mathbf{i}$, $\quad z_2' = 1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\mathbf{i}$

Hence, the required polynomial equation of degree six is

$$[z\text{-}(1+i)][z\text{-}(1\text{-}i)]\left[z\text{-}\left(1\text{-}\frac{\sqrt{3}}{2}\text{-}\frac{1}{2}i\right)\right]\left[z\text{-}\left(1\text{-}\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\right]\left[z\text{-}\left(1+\frac{\sqrt{3}}{2}\text{-}\frac{1}{2}i\right)\right]\left[z\text{-}\left(1+\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\right]=0 ,$$

which is the product of three quadratics:

$$(z^2 - 2z + 2)\left(z^2 + \sqrt{3}z - 2z - \sqrt{3} + 2\right)\left(z^2 - \sqrt{3}z - 2z + \sqrt{3} + 2\right) = 0$$
$$(z^2 - 2z + 2)(z^4 - 4z^3 + 5\,z^2 - 2z + 1) = 0$$
$$z^6 - 6\,z^5 + 15\,z^4 - 20\,z^3 + 15\,z^2 - 6\,z + 2 = 0$$

Method 2 (faster, but it does not use the former part of quadratics)

However, we you simply find a polynomial equation of degree six with real coefficients.

Consider $[(z-1)^3 - i^3][(z-1)^3 + i^3] = 0$,

which obviously has the roots of the equation $(z-1)^3 = i^3$.

$[(z-1)^3 + i][(z-1)^3 - i] = 0$ (note that the conjugates are also roots)

$(z-1)^6 - i^2 = 0$

$(z-1)^6 + 1 = 0$

$z^6 - 6z^5 + 15z^4 - 20z^3 + 15z^2 - 6z + 2 = 0$

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